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On the $|C, \alpha>1/2, \beta>1/2|$-summability of double orthogonal series

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Dedicated to Professor Károly Tandori on his 60th birthday

1. Introduction

Let $(X, \mathscr{F}, \mu)$ be a positive measure space and $\{\varphi_{ik}(x): i, k=0, 1, \ldots\}$ an orthonormal system (in abbreviation: ONS) on X. We will consider the double orthogonal series

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik} \varphi_{ik}(x) \tag{1.1}$$

where $\{a_{ik}: i, k=0, 1, \ldots\}$ is a sequence of real numbers (coefficients).

Let α and β be real numbers, $\alpha>-1$ and $\beta>-1$. We remind that the (C, α, β)-means of series (1.1) are defined by

$$\sigma_{mn}^{\alpha\beta}(x) = \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^\alpha A_{n-k}^\beta a_{ik} \varphi_{ik}(x)$$

where

$$A_m^\alpha = \binom{m+\alpha}{m} \quad (m, n = 0, 1, \ldots)$$

(for single series see, e.g. [13, p. 77]). Rhe case $\alpha=\beta=0$ gives back the rectangular partial sums:

$$\sigma_{mn}^{00}(x) = \sum_{i=0}^{m} \sum_{k=0}^{n} a_{ik} \varphi_{ik}(x) = s_{mn}(x),$$

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while the case $\alpha=\beta=1$ provides the first arithmetic means of the rectangular partial sums:

$$\sigma_{mn}^{11}(x) = \sum_{i=0}^{m}\sum_{k=0}^{n}\left(1-\frac{i}{m+1}\right)\left(1-\frac{k}{n+1}\right)a_{ik}\varphi_{ik}(x) =$$

$$= \frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n} s_{ik}(x) \quad (m, n = 0, 1, \ldots).$$

2. Main results

Series (1.1) is said to be absolute (C, α, β)-summable (in abbreviation: $|C, \alpha, \beta|$-summable) at a point x if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |\sigma_{mn}^{\alpha\beta}(x)-\sigma_{m-1,n}^{\alpha\beta}(x)-\sigma_{m,n-1}^{\alpha\beta}(x)+\sigma_{m-1,n-1}^{\alpha\beta}(x)| < \infty \tag{2.1}$$

where we agree on putting

$$\sigma_{-1,n}^{\alpha\beta}(x) = \sigma_{m,-1}^{\alpha\beta}(x) = \sigma_{-1,-1}^{\alpha\beta}(x) \equiv 0 \quad (m, n = 0, 1, \ldots). \tag{2.2}$$

In other words, the series in (2.1) can be rewritten as

$$|\sigma_{00}^{\alpha\beta}(x)| + \sum_{m=1}^{\infty} |\sigma_{m0}^{\alpha\beta}(x)-\sigma_{m-1,0}^{\alpha\beta}(x)| + \sum_{n=1}^{\infty} |\sigma_{0n}^{\alpha\beta}(x)-\sigma_{0,n-1}^{\alpha\beta}(x)|$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\sigma_{mn}^{\alpha\beta}(x)-\sigma_{m-1,n}^{\alpha\beta}(x)-\sigma_{m,n-1}^{\alpha\beta}(x)+\sigma_{m-1,n-1}^{\alpha\beta}(x)|.$$

In the sequel, we will use the notations

$$\Delta_{mn}^{\alpha\beta}(x) = \sigma_{mn}^{\alpha\beta}(x)-\sigma_{m-1,n}^{\alpha\beta}(x)-\sigma_{m,n-1}^{\alpha\beta}(x)+\sigma_{m-1,n-1}^{\alpha\beta}(x) \quad (m, n = 0, 1, \ldots) \tag{2.3}$$

with agreement (2.2), and

$$\mathscr{A}_{pq} = \Big\{\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\Big\}^{1/2} \quad (p, q = 0, 1, \ldots)$$

while identifying 2^{-1} with 0 in this paper.

Theorem 1. *If* $\alpha>1/2$, $\beta>1/2$, *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \mathscr{A}_{pq} < \infty, \tag{2.4}$$

then series (1.1) *is* $|C, \alpha, \beta|$*-summable a.e. on* X.

Theorem 1 is the extension of the theorems of Tandori [11] ($\alpha=1$) and Leindler [5] ($\alpha>1/2$) from single orthogonal series to double ones. In the special case of two-dimensional trigonometric series, Theorem 1 was proved by PONOMARENKO and TIMAN [8].

Condition (2.4) is not only sufficient but also necessary for the a.e. $|C, \alpha, \beta|$-summability of series (1.1), for a fixed pair of $\alpha>1/2$ and $\beta>1/2$, if all ONS $\{\varphi_{ik}(x)\}$ are considered.

To be more specific, let $(X, \mathscr{F}, \mu)$ be the unit square $[0, 1]\times[0, 1]$ with the Borel measurable subsets as $\mathscr{F}$ and with the plane Lebesgue measure as μ. In the sequel, the unit interval $[0, 1]$ is denoted by I, the unit square $I\times I$ by S, and the Lebesgue measure by $|\cdot|$ (it will be clear from the context whether $|\cdot|$ means the linear or plane measure). We consider the two-dimensional Rademacher system $\{r_i(x_1)r_k(x_2)$: $i, k=0, 1, \ldots\}$ on S, where the

$$r_i(x_1) = \operatorname{sign} \sin (2^i \pi x_1) \quad (i = 0, 1, \ldots;\ x_1 \in I)$$

are the well-known Rademacher functions (see, e.g. [1, p. 51] or [13, p. 212]).

Theorem 2. *If $\alpha>1/2$, $\beta>1/2$, and condition* (2.4) *is not satisfied, then the two-dimensional Rademacher series*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik} r_i(x_1) r_k(x_2) \tag{2.5}$$

is not $|C, \alpha, \beta|$-summable a.e. on S.

This theorem is the extension of the corresponding results of BILLARD [2] $(\alpha=1)$ and GREPAČEVSKAJA [4] $(\alpha>1/2)$ from single orthogonal series to double ones.

Putting Theorems 1 and 2 together, we can draw the next

Corollary 1. *Let $\alpha>1/2$ and $\beta>1/2$. Series* (1.1) *is $|C, \alpha, \beta|$-summable a.e. for every double* ONS $\{\varphi_{ik}(x)\colon x=(x_1, x_2)\}$ *defined on S if and only if condition* (2.4) *is satisfied.*

This result for single ONS defined on I was proved by TANDORI [11] and LEINDLER [5] $(\alpha>1/2)$. Both authors constructed a new ONS in the counterexample, rather than using the Rademacher system.

Let $\{\lambda_{ik}\colon i, k=0, 1, \ldots\}$ be a nondecreasing sequence of positive numbers such that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(i+1)(k+1)\lambda_{ik}} < \infty, \tag{2.6}$$

or equivalently,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{\lambda_{2^p, 2^q}} < \infty.$$

We note that $\{\lambda_{ik}\}$ is said to be nondecreasing if

$$\lambda_{ik} \leqq \min \{\lambda_{i+1,k},\ \lambda_{i,k+1}\} \quad (i, k = 0, 1, \ldots).$$

Applying the Cauchy inequality to the series

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{\lambda_{2^{p}-1,2^{q}-1}^{1/2}} \lambda_{2^{p}-1,2^{q}-1}^{1/2} \Big\{ \sum_{i=2^{p}-1}^{2^{p}-1} \sum_{k=2^{q}-1}^{2^{q}-1} a_{ik}^{2} \Big\}^{1/2}$$

yields the following

Corollary 2. *If $\alpha > 1/2$, $\beta > 1/2$, and $\{\lambda_{ik}\}$ is a nondecreasing sequence of positive numbers satisfying condition* (2.6) *and*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^{2} \lambda_{ik} < \infty,$$

then series (1.1) *is $|C, \alpha, \beta|$-summable a.e.*

The corresponding result for single orthogonal series is due to Ul'janov [12, pp. 46—47].

3. Generalizations and extensions

a) Let l be a real number, $l \geqq 1$. Following Flett [3], series (1.1) is said to be $|C, \alpha, \beta|_l$-summable at x if

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [(m+1)(n+1)]^{l-1} |\Delta_{mn}^{\alpha\beta}(x)|^{l} < \infty.$$

The case $l=1$ gives back $|C, \alpha, \beta|$-summability.

Using the same arguments as in the proof of Theorems 1 and 2, one can derive the following three generalizations.

Theorem 1a. *If $\alpha > 1/2$, $\beta > 1/2$, $1 \leqq l \leqq 2$, and*

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \mathscr{A}_{pq}^{l} < \infty, \tag{3.1}$$

then series (1.1) *is $|C, \alpha, \beta|_l$-summable a.e.*

Theorem 2a. *If $\alpha > 1/2$, $\beta > 1/2$, $1 \leqq l \leqq 2$, and condition* (3.1) *is not satisfied, then series* (2.5) *is not $|C, \alpha, \beta|_l$-summable a.e.*

Corollary 2a. *If $\alpha > 1/2$, $\beta > 1/2$, $1 \leqq l \leqq 2$, and $\{\lambda_{ik}\}$ is a nondecreasing sequence of positive numbers satisfying the conditions*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(i+1)(k+1)\lambda_{ik}^{l}} < \infty \tag{3.2}$$

and

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2 \lambda_{ik}^{2-l} < \infty,$$

then series (1.1) *is* $|C, \alpha, \beta|_l$*-summable a.e.*

In case $l=2$, condition (3.2) can be dropped.

Theorems 1a, 2a and Corollary 2a are the extensions of the corresponding theorems of SZALAY [9] and SPEVAKOV [10], respectively, from single orthogonal series to double ones.

b) Let $\varkappa=\{\varkappa_i\colon i=0, 1, \ldots\}$ and $\lambda=\{\lambda_k\colon k=0, 1, \ldots\}$ be two strictly increasing sequences of nonnegative numbers, both tending to ∞. Instead of (C, α, β)-means we can consider the $(R, \varkappa, \lambda)$-means of series (1.1) defined by

$$\sigma_{mn}(\varkappa, \lambda;\, x) = \sum_{i=0}^{m}\sum_{k=0}^{n}\left(1-\frac{\varkappa_i}{\varkappa_{m+1}}\right)\left(1-\frac{\lambda_k}{\lambda_{n+1}}\right) a_{ik}\varphi_{ik}(x) \quad (m, n = 0, 1, \ldots)$$

(for single series see, e.g. [1, p. 139]). Series (1.1) is said to be $|R, \varkappa, \lambda|$-summable at x if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |\Delta_{mn}(\varkappa, \lambda;\, x)| < \infty,$$

where $\Delta_{mn}(\varkappa, \lambda;\, x)$ is defined similarly to (2.3).

Let $\varkappa(t)$ and $\lambda(t)$ be strictly increasing functions of the nonnegative variable t (constructed from the corresponding sequences, e.g. by means of linear interpolation), for which $\varkappa(i)=\varkappa_i$ and $\lambda(k)=\lambda_k$ for all nonnegative integers i and k. Denote by $\mathscr{K}(t)$ and $\Lambda(t)$ the uniquely determined inverse functions of $\varkappa(t)$ and $\lambda(t)$, respectively. For the sake of brevity, we write

$$\eta_p = [\mathscr{K}(2^p)], \quad \nu_q = [\Lambda(2^q)],$$

and

$$\tilde{\mathscr{A}}_{pq} = \Big\{\sum_{i=\eta_{p-1}}^{\eta_p-1}\sum_{k=\nu_{q-1}}^{\nu_q-1} a_{ik}^2\Big\}^{1/2} \quad (p, q = 0, 1, \ldots;\ \eta_{-1} = \nu_{-1} = 0),$$

where $[\cdot]$ denotes the integral part and in case $\eta_{p-1}=\eta_p$ or $\nu_{q-1}=\nu_q$ we take $\tilde{\mathscr{A}}_{pq}=0$.

The next two theorems can be also proved by using the methods of Sections 4 and 6.

Theorem 1b. *If*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\tilde{\mathscr{A}}_{pq} < \infty, \tag{3.3}$$

then series (1.1) *is* $|R, \varkappa, \lambda|$*-summable a.e.*

Theorem 2b. *If condition* (3.3) *is not satisfied, then series* (2.5) *is not* $|R, \varkappa, \lambda|$-*summable a.e.*

Both theorems are the extensions of those in [6] by the present author from single orthogonal series to double ones.

c) Finally, we point out that Theorems 1 and 2 as well as their variations mentioned above in (a) and (b) admit quite natural extensions to multiple orthogonal series, too.

4. Proof of Theorem 1

One important thing is to find a good representation for $\Delta_{mn}^{\alpha\beta}(x)$ defined by (2.3). Using the identities

$$A_{m-1}^{\alpha} = \frac{m}{\alpha+m} A_m^{\alpha}, \quad A_{m-i}^{\alpha} = \frac{\alpha+m-i}{\alpha} A_{m-i}^{\alpha-1},$$

and

$$A_{m-i-1}^{\alpha} = \frac{m-i}{\alpha} A_{m-i}^{\alpha-1} \quad (\alpha > -1, \quad \alpha \neq 0),$$

one can obtain for $m \geqq 1$ and $n \geqq 1$

$$\Delta_{mn}^{\alpha\beta}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}} \frac{A_{n-k}^{\beta-1}}{A_n^{\beta}} \frac{ik}{mn} a_{ik} \varphi_{ik}(x), \tag{4.1}$$

for $m \geqq 1$ and $n=0$

$$\Delta_{m0}^{\alpha\beta}(x) = \sum_{i=1}^{m} \frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}} \frac{i}{m} a_{i0} \varphi_{i0}(x), \tag{4.2}$$

for $m=0$ and $n \geqq 1$

$$\Delta_{0n}^{\alpha\beta}(x) = \sum_{k=1}^{n} \frac{A_{n-k}^{\beta-1}}{A_n^{\beta}} \frac{k}{n} a_{0k} \varphi_{0k}(x), \tag{4.3}$$

while for $m=n=0$, $\Delta_{00}^{\alpha\beta}(x) = \sigma_{00}(x) = a_{00}\varphi_{00}(x)$. These representations are valid even in the cases where $\alpha=0$ or $\beta=0$, i.e. for all values of $\alpha > -1$ and $\beta > -1$.

We recall the inequality

$$\sum_{m=1}^{\infty} \left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2 = O\left\{\frac{1}{i}\right\} \quad (i = 1, 2, \ldots; \alpha > 1/2), \tag{4.4}$$

which is well-known in the literature (see, e.g. [1, p. 110]).

By Minkowski's inequality (keeping agreement (2.2) in mind),

$$\left\{\int_X \left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\Delta_{mn}^{\alpha\beta}(x)|\right]^2 d\mu(x)\right\}^{1/2} \leqq \left\{\int_X [\sigma_{00}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2}+ \tag{4.5}$$

$$+\sum_{m=1}^{\infty}\left\{\int_X [\Delta_{m0}^{\alpha\beta}(x)]^2 d\mu(x)\right\}+\sum_{n=1}^{\infty}\left\{\int_X [\Delta_{0n}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2}+$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\int_X [\Delta_{mn}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2} = S_1+S_2+S_3+S_4, \quad \text{say.}$$

Obviously, $S_1=|a_{00}|=\mathscr{A}_{00}$. We are going to show that S_2, S_3, and S_4 are also finite. We treat S_4 in more detail.

Applying the Cauchy inequality, then (4.1) and (4.4), we get that

$$S_4=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\left\{\int_X [\Delta_{mn}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2} \leqq \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{p/2}2^{q/2}\times$$

$$\times\left\{\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\int_X [\Delta_{mn}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{p/2}2^{q/2}\times$$

$$\times\left\{\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\sum_{i=1}^{m}\sum_{k=1}^{n}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\right]^2\left[\frac{A_{n-k}^{\beta-1}}{A_n^\beta}\right]^2\frac{i^2k^2}{m^2n^2}a_{ik}^2\right\}^{1/2} \leqq$$

$$\leqq \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p/2}2^{-q/2}\left\{\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\sum_{i=1}^{m}\sum_{k=1}^{n}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\right]^2\left[\frac{A_{n-k}^{\beta-1}}{A_n^\beta}\right]^2 i^2k^2a_{ik}^2\right\}^{1/2}=$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p/2}2^{-q/2}\left\{\sum_{i=1}^{2^{p+1}-1}\sum_{k=1}^{2^{q+1}-1}i^2k^2a_{ik}^2\sum_{m=\max\{2^p,i\}}^{2^{p+1}-1}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\right]^2\sum_{n=\max\{2^q,k\}}^{2^{q+1}-1}\left[\frac{A_{n-k}^{\beta-1}}{A_n^\beta}\right]^2\right\}^{1/2}=$$

$$=O\{1\}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p}2^{-q}\left\{\sum_{i=1}^{2^{p+1}-1}\sum_{k=1}^{2^{q+1}-1}i^2k^2a_{ik}^2\right\}^{1/2}.$$

Finally, using the elementary inequality

$$\{a+b+\ldots\}^{1/2}\leqq a^{1/2}+b^{1/2}+\ldots \quad (a\geqq 0,\quad b\geqq 0,\ldots)$$

yields

$$\begin{aligned} S_4 &= O\{1\}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p}2^{-q}\left\{\sum_{j=1}^{p+1}\sum_{l=1}^{q+1}2^{2j}2^{2l}\mathscr{A}_{jl}^2\right\}^{1/2}= \\ &= O\{1\}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p}2^{-q}\sum_{j=1}^{p+1}\sum_{l=1}^{q+1}2^j2^l\mathscr{A}_{jl}= \\ &= O\{1\}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}2^j2^l\mathscr{A}_{jl}\sum_{p=j-1}^{\infty}2^{-p}\sum_{q=l-1}^{\infty}2^{-q}= \\ &= O\{1\}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\mathscr{A}_{jl}. \end{aligned} \tag{4.6}$$

Following the above pattern, one can show that

$$S_2 = O\{1\} \sum_{j=1}^{\infty} \mathscr{A}_{j0} \quad \text{and} \quad S_3 = O\{1\} \sum_{l=1}^{\infty} \mathscr{A}_{0l}. \tag{4.7}$$

Collecting (4.5)—(4.7) together, by (2.4) we can conclude

$$\Big\{\int_X \Big[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}^{\alpha\beta}(x)|\Big]^2 d\mu(x)\Big\}^{1/2} = O\{1\} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \mathscr{A}_{jl} < \infty.$$

This means that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}^{\alpha\beta}(x)| \in L^2(X, \mathscr{F}, \mu)$$

and, in particular, (2.1) follows for almost every x.

5. Auxiliary results on finite sums in terms of two-dimensional Rademacher functions

We remind the following two results concerning finite sums in terms of one-dimensional Rademacher functions.

Lemma A. (see, e.g. [13, p. 213, Theorem (8.4)]). *For every $r>0$ there exists a constant D_r such that for every finite sum*

$$P(x_1) = \sum_{k=0}^{N} a_k r_k(x_1) \quad (N = 0, 1, \ldots)$$

we have

$$\int_0^1 |P(x_1)|^r dx_1 \leqq D_r \Big\{\sum_{k=0}^{N} a_k^2\Big\}^{r/2}. \tag{5.1}$$

Lemma B. (Orlicz [7]). *Given any measurable set $E(\subset I)$ of positive measure, there exist an integer k_0 and a constant $C_1>0$ such that for every finite sum*

$$P(x_1) = \sum_{k=k_0}^{N} a_k r_k(x_1) \quad (N \geqq k_0)$$

we have

$$\int_E |P(x_1)| dx_1 \geqq C_1 \Big\{\sum_{k=k_0}^{N} a_k^2\Big\}^{1/2}.$$

Our goal is to extend these results to finite sums in terms of two-dimensional Rademacher functions. We note that the extension to higher dimensions runs in the same way.

Lemma 1. *For every finite sum*

$$P(x_1, x_2) = \sum_{i=0}^{M} \sum_{k=0}^{N} a_{ik} r_i(x_1) r_k(x_2) \quad (M, N = 0, 1, \ldots)$$

we have

$$\int_0^1 \int_0^1 [P(x_1, x_2)]^4 dx_1 dx_2 \leqq D_4^2 [\sum_{i=0}^{M} \sum_{k=0}^{N} a_{ik}^2]^2. \tag{5.2}$$

Lemma 1 actually holds true for every $r>0$ instead of $r=4$ (cf. the proof of Lemma A in [13, pp. 213—214]). But inequality (5.2) is enough for our purpose.

Proof. By Fubini's theorem and (5.1) for the $r_i(x_1)$,

$$J = \int_0^1 \int_0^1 [P(x_1, x_2)]^4 dx_1 dx_2 = \int_0^1 \Big\{\int_0^1 [\sum_{i=0}^{M} (\sum_{k=0}^{N} a_{ik} r_k(x_2)) r_i(x_1)]^4 dx_1\Big\} dx_2 \leqq$$

$$\leqq D_4 \int_0^1 [\sum_{i=0}^{M} [\sum_{k=0}^{N} a_{ik} r_k(x_2)]^2]^2 dx_2 = D_4 \sum_{i=0}^{M} \int_0^1 [\sum_{k=0}^{N} a_{ik} r_k(x_2)]^4 dx_2 +$$

$$+ 2D_4 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M} \int_0^1 [\sum_{k=0}^{N} a_{ik} r_k(x_2)]^2 [\sum_{k=0}^{N} a_{jk} r_k(x_2)]^2 dx_2.$$

Applying the Cauchy—Schwarz inequality and (5.1) for the $r_k(x_2)$,

$$J \leqq D_4^2 \sum_{i=0}^{M} [\sum_{k=0}^{N} a_{ik}^2]^2 +$$

$$+ 2D_4 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M} \Big\{\int_0^1 [\sum_{k=0}^{N} a_{ik} r_k(x_2)]^4 dx_2 \int_0^1 [\sum_{k=0}^{N} a_{jk} r_k(x_2)]^4 dx_2\Big\}^{1/2} \leqq$$

$$\leqq D_4^2 \sum_{i=0}^{M} [\sum_{k=0}^{N} a_{ik}^2]^2 + 2D_4^2 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M} [\sum_{k=0}^{N} a_{ik}^2][\sum_{k=0}^{N} a_{jk}^2] = D_4^2 [\sum_{i=0}^{M} \sum_{k=0}^{N} a_{ik}^2]^2.$$

Lemma 2. *Given any measurable set $E(\subset S)$ of positive measure, there exist an integer n_0 and a constant $C_2>0$ such that for every finite sum*

$$P(x_1, x_2) = \sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik} r_i(x_1) r_k(x_2) \quad \textit{with} \quad \max\{m, n\} \geqq n_0 \tag{5.3}$$

$$(M \geqq m \geqq 0 \quad \textit{and} \quad N \geqq n \geqq 0),$$

we have

$$\iint_E |P(x_1, x_2)| dx_1 dx_2 \geqq C_2 \{\sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik}^2\}^{1/2}. \tag{5.4}$$

Proof. We consider the 4-tuple system Π_4 defined by

$$\Pi_4 = \{r_{i_1}(x_1)r_{k_1}(x_2)r_{i_2}(x_1)r_{k_2}(x_2)\colon\ i_1, i_2, k_1, k_2 = 0, 1, \ldots;\quad i_1 \leqq i_2,\ k_1 \leqq k_2, \text{ and } (i_1, k_1) \neq (i_2, k_2)\}.$$

The proof of Lemma B (see [7]) is based on the fact that the 2-tuple system

$$\Pi_2 = \{r_{i_1}(x_1)r_{i_2}(x_1)\colon\ i_1, i_2 = 0, 1, \ldots \text{ and } i_1 < i_2\}$$

is an ONS on I. Hence it immediately follows that Π_4 is an ONS on S.

By Bessel's inequality, for any function $f(x_1, x_2) \in L^2(S)$,

$$\sum_{i_1=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{\substack{i_2=i_1\\(i_1, k_1)\neq(i_2,k_2)}}^{\infty}\sum_{k_2=k_1}^{\infty}\Big[\int_0^1\int_0^1 f(x_1, x_2)r_{i_1}(x_1)r_{k_1}(x_2)\,r_{i_2}(x_1)r_{k_2}(x_2)\,dx_1\,dx_2\Big]^2 < \infty.$$

Letting $f=\chi_E$, the characteristic function of the set E, and $\varepsilon=|E|^2/16$ there exists an integer n_0 such that

$$\sum_{i_1=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{\substack{i_2=i_1\\(i_1,k_1)\neq(i_2,k_2)\\ \max\{i_1,k_1\}\geqq n_0}}^{\infty}\sum_{k_2=k_1}^{\infty}\Big[\iint_E r_{i_1}(x_1)r_{k_1}(x_2)r_{i_2}(x_1)r_{k_2}(x_2)\,dx_1\,dx_2\Big]^2 \leqq \varepsilon. \tag{5.5}$$

We consider a finite sum $P(x_1, x_2)$ given by (5.3). Applying Hölder's inequality with the exponents 3/2 and 3, while representing 2 as the sum of 2/3 and 4/3, we get that

$$\iint_E [P(x_1, x_2)]^2\,dx_1\,dx_2 \leqq \Big\{\iint_E |P(x_1, x_2)|\,dx_1\,dx_2\Big\}^{2/3}\Big\{\iint_E [P(x_1, x_2)]\,dx_1\,dx_2\Big\}^{1/3},$$

whence

$$\iint_E |P(x_1, x_2)|\,dx_1\,dx_2 \geqq \Big\{\int_0^1\int_0^1 [P(x_1, x_2)]^4\,dx_1\,dx_2\Big\}^{-1/2}\Big\{\iint_E [P(x_1, x_2)]^2\,dx_1\,dx_2\Big\}^{3/2}. \tag{5.6}$$

We square out in $[P(x_1, x_2)]^2$ and use the Cauchy inequality and (5.5) to obtain

$$\iint_E [P(x_1, x_2)]^2\,dx_1\,dx_2 = |E|\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2 +$$

$$+\sum_{i_1=m}^{M}\sum_{\substack{k_1=n\\(i_1,k_1)\neq(i_2,k_2)}}^{N}\sum_{i_2=m}^{M}\sum_{k_2=n}^{N} a_{i_1k_1}a_{i_2k_2}\iint_E r_{i_1}(x_1)r_{k_1}(x_2)r_{i_2}(x_1)r_{k_2}(x_2)\,dx_1\,dx_2 \geqq$$

$$\geqq |E|\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2 - \Big\{\sum_{i_1}\sum_{k_1}\sum_{i_2}\sum_{k_2} a_{i_1k_1}^2 a_{i_2k_2}^2 \times$$

$$\times \sum_{i_1}\sum_{k_1}\sum_{i_2}\sum_{k_2}\Big[\iint_E r_{i_1}(x_1)r_{k_1}(x_2)r_{i_2}(x_1)r_{k_2}(x_2)\,dx_1\,dx_2\Big]^2\Big\}^{1/2} \geqq$$

$$\geqq |E|\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2 - \{4\varepsilon\}^{1/2}\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2 = \frac{|E|}{2}\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2.$$

On the other hand, taking (5.2) into account, from (5.6) we get that

$$\iint_E |P(x_1, x_2)|\,dx_1 dx_2 \geqq \frac{|E|^{3/2}}{2^{3/2} D_4} \Big\{\sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik}^2\Big\}^{1/2},$$

which is (5.3) to be proved.

6. Proof of Theorem 2

We need the following properties of the binomial coefficients.

(i) A_m^α is positive for $\alpha>-1$, and is increasing as a function of m for $\alpha>0$ (see, e.g. [13, p. 77, Theorem (1.17)]).

(ii) There exist two positive constants C_3 and C_4 depending only on α such that

$$C_3 \leqq \frac{A_m^\alpha}{m^\alpha} \leqq C_4 \quad (m=1, 2, \ldots;\ \alpha>-1)$$

(see, e.g. [1, p. 69, formula (25)]). Hence one can derive that

$$\sum_{m=i}^{\infty} \frac{A_{m-i}^{\alpha-1}}{m A_m^\alpha} = O\{1\} \quad (i=0, 1, \ldots;\ \alpha>0). \tag{6.1}$$

(iii) There exists a positive constant C_5 depending on α such that

$$\frac{A_{2p}^\alpha}{A_{2p+1}^\alpha} \geqq C_5 \quad (p=0, 1, \ldots;\ \alpha>0). \tag{6.2}$$

We will prove that if series (2.5) is $|C, \alpha, \beta|$-summable on some subset of S with positive measure, for a certain pair of $\alpha>1/2$ and $\beta>1/2$, then (2.4) necessarily holds. Consequently, if (2.4) is not satisfied, then series (2.5) can be $|C, \alpha, \beta|$-summable only on a set of measure zero for any pair of $\alpha>1/2$ and $\beta>1/2$.

To begin with, by Egorov's theorem there exist a constant B and a set $E(\subset S)$ of positive measure such that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| \leqq B \quad \text{for every } (x_1, x_2) \in E. \tag{6.3}$$

We are going to apply Lemma 2. To this end, we must get rid of the Rademacher functions $r_i(x_1) r_k(x_2)$ in the definition of $\Delta_{mn}^{\alpha\beta}(x_1, x_2)$ for which $\max\{i, k\}<n_0$. This can be done in the following way. Set temporarily

$$\tilde{a}_{ik} = \begin{cases} a_{ik} & \text{if } \max\{i, k\} \geqq n_0, \\ 0 & \text{if } \max\{i, k\} < n_0; \end{cases}$$

and denote by $\tilde{\Delta}_{mn}^{\alpha\beta}(x_1, x_2)$ the corresponding differences. Then, by (6.1)

$$\sum_{\substack{m=1\\ \max\{m,n\}\geqq n_0}}^{\infty}\sum_{n=1}^{\infty} |\tilde{\Delta}_{mn}^{\alpha\beta}(x_1, x_2)| \leqq \sum_{\substack{m=1\\ \max\{m,n\}\geqq n_0}}^{\infty}\sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| +$$

$$+ \sum_{\substack{m=1\\ \max\{m,n\}\geqq n_0}}^{\infty}\sum_{n=1}^{\infty} \sum_{i=1}^{\min\{m,n_0-1\}} \sum_{k=1}^{\min\{n,n_0-1\}} \frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}} \frac{A_{n-k}^{\beta-1}}{A_n^{\beta}} \frac{ik}{mn} |a_{ik}| =$$

$$= \sum_{\substack{m=1\\ \max\{m,n\}\geqq n_0}}^{\infty}\sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| + \sum_{i=1}^{n_0-1}\sum_{k=0}^{n_0-1} ik|a_{ik}| \Big\{ \sum_{m=i}^{n_0-1}\sum_{n=n_0}^{\infty} +$$

$$+ \sum_{m=n_0}^{\infty}\sum_{n=n_0}^{\infty} + \sum_{m=n_0}^{\infty}\sum_{n=k}^{n_0-1} \Big\} \frac{A_{m-i}^{\alpha-1}}{mA_m^{\alpha}} \frac{A_{n-k}^{\beta-1}}{nA_n^{\beta}} \leqq$$

$$\leqq \sum_{\substack{m=1\\ \max\{m,n\}\geqq n_0}}^{\infty}\sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| + 3 \sum_{i=1}^{n_0-1}\sum_{k=1}^{n_0-1} ik|a_{ik}| \times$$

$$\times \sum_{m=i}^{\infty} \frac{A_{n-k}^{\alpha-1}}{mA_m^{\alpha}} \sum_{n=k}^{\infty} \frac{A_{n-k}^{\beta-1}}{nA_n^{\beta}} = \sum_{\substack{m=1\\ \max\{m,n\}\geqq n_0}}^{\infty}\sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| + O\{1\},$$

where $O\{1\}$ does not depend on (x_1, x_2). One can similarly obtain that

$$\sum_{m=n_0}^{\infty} |\tilde{\Delta}_{m0}^{\alpha\beta}(x_1, x_2)| = \sum_{m=n_0}^{\infty} |\Delta_{m0}^{\alpha\beta}(x_1, x_2)| + O\{1\}$$

and

$$\sum_{n=n_0}^{\infty} |\tilde{\Delta}_{0n}^{\alpha\beta}(x_1, x_2)| = \sum_{n=n_0}^{\infty} |\Delta_{0n}^{\alpha\beta}(x_1, x_2)| + O\{1\}.$$

So we may assume, without loss of generality, that $a_{ik}=0$ for $i, k=0, 1, \ldots, \ldots, n_0-1$ and use the notations a_{ik} and $\Delta_{mn}^{\alpha\beta}(x_1, x_2)$, rather than $\tilde{a}_{ik}$ and $\tilde{\Delta}_{mn}^{\alpha\beta}(x_1, x_2)$. Set

$$\delta_{pq}^{\alpha\beta}(x_1, x_2) = \sigma_{2^{p+1}-1, 2^{q+1}-1}^{\alpha\beta}(x_1, x_2) - \sigma_{2^{p-1}-1, 2^{q+1}-1}^{\alpha\beta}(x_1, x_2) -$$

$$- \sigma_{2^{p+1}-1, 2^{q-1}-1}^{\alpha\beta}(x_1, x_2) + \sigma_{2^{p-1}-1, 2^{q-1}-1}^{\alpha\beta}(x_1, x_2) \quad (p, q = 1, 2, \ldots).$$

On the one hand,

$$|\delta_{pq}^{\alpha\beta}(x_1, x_2)| \leqq \sum_{m=2^{p-1}}^{2^{p+1}-1} \sum_{n=2^{q-1}}^{2^{q+1}-1} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)|,$$

whence

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} |\delta_{pq}^{\alpha\beta}(x_1, x_2)| \leqq 4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)|.$$

Consequently, by (6.3)

$$\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \iint_E |\delta_{pq}^{\alpha\beta}(x_1, x_2)| \, dx_1 dx_2 \leqq 4B|E|. \tag{6.4}$$

On the other hand, by (5.4)

$$\iint_E |\delta_{pq}^{\alpha\beta}(x_1, x_2)| \, dx_1 dx_2 \geqq \tag{6.5}$$

$$\geqq C_2 \left\{ \sum_{i=2^{p-1}}^{2^{p+1}-1} \sum_{k=2^{q-1}}^{2^{q+1}-1} \left[\frac{A_{2^{p+1}-1-i}^{\alpha}}{A_{2^{p+1}-1}^{\alpha}} \right]^2 \left[\frac{A_{2^{q+1}-1-k}^{\beta}}{A_{2^{q+1}-1}^{\beta}} \right]^2 a_{ik}^2 \right\}^{1/2}.$$

On applying Lemma 2, we took into consideration only those terms in the representation of $\delta_{pq}^{\alpha\beta}(x_1, x_2)$ which contain a_{ik} with $2^{p-1} \leqq i \leqq 2^{p+1}-1$ and $2^{q-1} \leqq k \leqq 2^{q+1}-1$. Due to the monotony of A_m^{α} in m (see (ii) at the beginning at this Section) and (6.2) for $2^{p-1} \leqq i \leqq 2^p - 1$ and $2^{q-1} \leqq k \leqq 2^q - 1$,

$$\frac{A_{2^{p+1}-1-i}^{\alpha}}{A_{2^{p+1}-1}^{\alpha}} \frac{A_{2^{q+1}-1-k}^{\beta}}{A_{2^{q+1}-1}^{\beta}} \geqq \frac{A_{2^p}^{\alpha}}{A_{2^{p+1}}^{\alpha}} \frac{A_{2^q}^{\beta}}{A_{2^{q+1}}^{\beta}} \geqq C_5^2.$$

From here and (6.5) it follows that

(6.6)

$$\iint_E |\delta_{pq}^{\alpha\beta}(x_1, x_2)| \, dx_1 dx_2 \geqq C_2 C_5^2 \Big\{ \sum_{i=2^{p-1}}^{2^p-1} \sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2 \Big\}^{1/2} = C_2 C_5^2 \mathscr{A}_{pq} \quad (p, q = 1, 2, \ldots).$$

Combining (6.4) and (6.6) yields

$$\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \mathscr{A}_{pq} \leqq \frac{4B|E|}{C_2 C_5^2} < \infty. \tag{6.7}$$

Starting with

$$\delta_{p0}^{\alpha\beta}(x_1, x_2) = \sigma_{2^{p+1}-1,0}^{\alpha\beta}(x_1, x_2) - \sigma_{2^{p-1}-1,0}^{\alpha\beta}(x_1, x_2) \quad (p = 1, 2, \ldots)$$

and

$$\delta_{0q}^{\alpha\beta}(x_1, x_2) = \sigma_{0,2^{q+1}-1}^{\alpha\beta}(x_1, x_2) - \sigma_{0,2^{q-1}-1}^{\alpha\beta}(x_1, x_2) \quad (q = 1, 2, \ldots),$$

respectively, one can find in the same manner that

$$\sum_{p=1}^{\infty} \mathscr{A}_{p0} < \infty \quad \text{and} \quad \sum_{q=1}^{\infty} \mathscr{A}_{0q} < \infty. \tag{6.8}$$

The fulfilment of (6.7) and (6.8) is equivalent to that of (2.4).

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